J. Myanmar Acad. Arts Sci. 2023 **Vol. XXII. No.2**

APPLYING ONE-DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD TO PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, definitions of one-dimensional differential transform (reduced differential transform) and inverse differential transform are described. And then, properties of one-dimensional differential transform are expressed. After that, one-dimensional differential transform is utilized to solve the initial value problems for linear one-dimensional partial differential equations with constant coefficients and variable coefficients. Finally, one-dimensional differential transform is applied to solve the initial value problems for two-dimensional second-order partial differential equations.

Keywords: partial differential equations, one-dimensional differential transform, applications

Introduction

This paper is a continuation of our previous work [Hnin and Khin, 2023]. In [Hnin and Khin, 2023], the one-dimensional differential transform method was used for solving initial value problems for ordinary differential equations with constant coefficients and variable coefficients. Differential transform method can be used for solving initial value problems for differential equations and integral equations. In this paper, we are interested in solving one-dimensional and two-dimensional second-order partial differential equations by using a one-dimensional differential transform method.

The rest of this paper is organized as follows: In Section 2, basic concepts of one-dimensional differential transforms are recalled. The main results are demonstrated in Section 3 and Section 4 respectively by solving the initial value problems for one-dimensional partial differential equations and two-dimensional partial differential equations using one-dimensional differential transform method.

Preliminaries

In this section, the definitions of one-dimensional differential transform and its properties are described.

Definition 1[Raslan, Biswas and Abu Sheer, 2012]

If $u(x,t)$ is analytic and differentiable continuously in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=t_0}, \tag{1}$$

where the spectrum $U_k(x)$ is the **transformed function**, which is called **T**-function.

Definition 2 [Raslan, Biswas and Abu Sheer, 2012]

Differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^k. \tag{2}$$

Substitution (1) into (2), we obtain

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* Special Award(2023)

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$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=t_0}(t-t_0)^k. \tag{3}$$

When (t_0) are taken as $(t_0=0)$, then (3) is expressed as

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=t_0}t^k,$$

and (1) is shown as

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)\,t^k. \tag{4}$$

In real application, the function $u(x,t)$ by a finite series of (4) can be written as

$$u(x,t)=\sum_{k=0}^{n}U_k(x)\,t^k, \tag{5}$$

usually, the value of n is decided by convergence of the series coefficients.

Some Properties of One-Dimensional Differential Transform

Theorem 1 [Khatib, (2016)]

If $z(x,t)=\alpha\,u(x,t)+\beta\,v(x,t),$ the differential transform of $z(x,t)$ be $Z_k(x),$ then $Z_k(x)=\alpha\,U_k(x)+\beta\,V_k(x),$ where α and β are constants.

Proof: Let $z(x,t)=\alpha\,u(x,t)+\beta\,v(x,t).$ Then,

$$\begin{aligned}
Z_k(x)&=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}z(x,t)\right]_{t=0}\\
&=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}(\alpha\,u(x,t)+\beta\,v(x,t))\right]_{t=0}\\
&=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}(\alpha\,u(x,t))+\frac{\partial^k}{\partial t^k}(\beta\,v(x,t))\right]_{t=0}\\
&=\frac{\alpha}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}+\frac{\beta}{k!}\left[\frac{\partial^k}{\partial t^k}v(x,t)\right]_{t=0}\\
&=\alpha\,U_k(x)+\beta\,V_k(x).
\end{aligned}$$

Therefore, $Z_k(x)=\alpha\,U_k(x)+\beta\,V_k(x).$ □

Theorem 2 [Khatib, (2016)]

If $z(x,t)=\frac{\partial}{\partial x}u(x,t),$ and the differential transform of $z(x,t)$ be $Z_k(x),$ then $Z_k(x)=\frac{\partial}{\partial x}U_k(x).$

Proof: Let $z(x,t)=\frac{\partial}{\partial x}u(x,t).$ Then, we have

$$Z_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}z(x,t)\right]_{t=0}$$

$$=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}(\frac{\partial}{\partial x}u(x,t))\right]_{t=0}$$

$$=\frac{\partial}{\partial x}\left[\frac{1}{k!}\left(\frac{\partial^k}{\partial t^k}u(x,t)\right)\right]_{t=0}$$

$$=\frac{\partial}{\partial x}U_k(x).$$

Therefore, $Z_k(x)=\frac{\partial}{\partial x}U_k(x).$ □

Theorem 3 [Khatib, (2016)]

If $z(x,t)=\frac{\partial^2}{\partial x^2}u(x,t),$ and the differential transform of $z(x,t)$ be $Z_k(x),$ then $Z_k(x)=\frac{\partial^2}{\partial x^2}U_k(x).$

Proof: Let $z(x,t)=\frac{\partial^2}{\partial x^2}u(x,t).$ Then, we have

$$Z_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}z(x,t)\right]_{t=0}$$

$$=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}(\frac{\partial^2}{\partial x^2}u(x,t))\right]_{t=0}$$

$$=\frac{\partial^2}{\partial x^2}\left[\frac{1}{k!}\left(\frac{\partial^k}{\partial t^k}u(x,t)\right)\right]_{t=0}$$

$$=\frac{\partial^2}{\partial x^2}U_k(x).$$

Therefore, $Z_k(x)=\frac{\partial^2}{\partial x^2}U_k(x).$ □

Theorem 4 [Khatib, (2016)]

If $z(x,t)=\frac{\partial^m}{\partial t^m}u(x,t),$ and the differential transform of $z(x,t)$ be $Z_k(x),$ then $Z_k(x)=\frac{(k+m)!}{k!}U_{k+m}(x).$

Proof: Let $z(x,t)=\frac{\partial^m}{\partial t^m}u(x,t).$ Then, we have

$$Z_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} z(x,t)\right]_{t=0}$$

$$= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\left(\frac{\partial^m}{\partial t^m} u(x,t)\right)\right]_{t=0}$$

$$= \frac{(k+m)!}{k!(k+m)!}\left[\frac{\partial^{k+m}}{\partial t^{k+m}} u(x,t)\right]_{t=0}$$

$$= \frac{(k+m)!}{k!} U_{k+m}(x).$$

Therefore, $Z_k(x) = \dfrac{(k+m)!}{k!} U_{k+m}(x).$ □

Theorem 5 (Leibniz's theorem)

If $y = uv$, where u and v are any functions of x, then

$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_r u_{n-r} v_r + \ldots + u v_n$, where, suffixes of u and v denote the number of times they are differentiated.

Proof: [See, Kishan, 2007]. □

Theorem 6 [Khatib, (2016)]

If $z(x,t) = u(x,t)\,v(x,t)$, and the differential transform of $z(x,t)$ be $Z_k(x)$, then $Z_k(x) = \sum_{r=0}^{k} U_r(x) V_{k-r}(x).$

Proof: Let $z(x,t) = u(x,t)\,v(x,t).$ Then,

$$Z_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} z(x,t)\right]_{t=0}$$

$$= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}(u(x,t)\,v(x,t))\right]_{t=0}.$$

Now, Leibnitz's theorem for partial derivatives of function of several variables

$$\frac{\partial^k}{\partial t^k}(u(x,t)\,v(x,t)) = \sum_{r=0}^{k}\binom{k}{r}\frac{\partial^r}{\partial t^r} u(x,t)\,\frac{\partial^{k-r}}{\partial t^{k-r}} v(x,t).$$

Then, we get

$$Z_k(x) = \frac{1}{k!}\left[\sum_{r=0}^{k}\binom{k}{r}\frac{\partial^r}{\partial t^r} u(x,t)\bigg|_{t=0}\,\frac{\partial^{k-r}}{\partial t^{k-r}} v(x,t)\bigg|_{t=0}\right]$$

$$= \frac{1}{r!(k-r)!}\left[\sum_{r=0}^{k}\frac{\partial^r}{\partial t^r} u(x,t)\bigg|_{t=0}\,\frac{\partial^{k-r}}{\partial t^{k-r}} v(x,t)\bigg|_{t=0}\right]$$

$$= \sum_{r=0}^{k} U_r(x)\,V_{k-r}(x).$$

Therefore, $Z_k(x) = \sum_{r=0}^{k} U_r(x) V_{k-r}(x).$ $\square$

Now we can extend definitions and theorems of differential transform and inverse differential transform for solving two-dimensional heat and wave equations.

Definition 3

If $u(x, y, t)$ is analytic and differentiable continuously in the domain of interest, then let

$$U_k(x, y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x, y, t)\right]_{t=t_0}, \tag{6}$$

where the spectrum $U_k(x, y)$ is the **transformed function**, which is called **T**-function.

Definition 4

Differential inverse transform of $U_k(x, y)$ is defined as follows:

$$u(x, y, t) = \sum_{k=0}^{\infty} U_k(x, y)(t - t_0)^k. \tag{7}$$

Substitution (6) into (7), we obtain

$$u(x, y, t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x, y, t)}{\partial t^k}\right]_{t=t_0} (t - t_0)^k. \tag{8}$$

When (t_0) are taken as $(t_0 = 0)$, then (8) is expressed as

$$u(x, y, t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x, y, t)}{\partial t^k}\right]_{t=t_0} t^k,$$

and (6) is shown as

$$u(x, y, t) = \sum_{k=0}^{\infty} U_k(x, y)\, t^k. \tag{9}$$

In real application, the function $u(x, y, t)$ by a finite series of (9) can be written as

$$u(x, y, t) = \sum_{k=0}^{n} U_k(x, y)\, t^k, \tag{10}$$

usually, the value of n is decided by convergence of the series coefficients.

Theorem 7

If $z(x, y, t) = \alpha u(x, y, t) + \beta v(x, y, t),$ then $Z_k(x, y) = \alpha U_k(x, y) + \beta V_k(x, y),$ where α and β are constants.

Proof: Let $z(x, y, t) = \alpha u(x, y, t) + \beta v(x, y, t)$ and the differential transform of $z(x, y, t)$ be $Z_k(x, y).$ Then,

$$Z_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} z(x,y,t)\right]_{t=0}$$

$$= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}(\alpha\, u(x,y,t) + \beta\ v(x,y,t))\right]_{t=0}$$

$$= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}(\alpha\, u(x,y,t)) + \frac{\partial^k}{\partial t^k}(\beta\, v(x,y,t))\right]_{t=0}$$

$$= \frac{\alpha}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,y,t)\right]_{t=0} + \frac{\beta}{k!}\left[\frac{\partial^k}{\partial t^k} v(x,y,t)\right]_{t=0}$$

$$= \alpha\, U_k(x,y) + \beta\, V_k(x,y).$$

Therefore, $Z_k(x,y) = \alpha\, U_k(x,y) + \beta\, V_k(x,y)$.

Theorem 8

If $z(x,y,t) = \frac{\partial}{\partial x} u(x,y,t)$, then $Z_k(x,y) = \frac{\partial}{\partial x} U_k(x,y)$.

Proof: Let $z(x,y,t) = \frac{\partial}{\partial x} u(x,y,t)$, and the differential transform of $z(x,y,t)$ be $Z_k(x,y)$. Then,

$$Z_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} z(x,y,t)\right]_{t=0}$$

$$= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}(\frac{\partial}{\partial x} u(x,y,t))\right]_{t=0}$$

$$= \frac{\partial}{\partial x}\left[\frac{1}{k!}\left(\frac{\partial^k}{\partial t^k} u(x,y,t)\right)\right]_{t=0} = \frac{\partial}{\partial x} U_k(x,y).$$

Therefore, $Z_k(x,y) = \frac{\partial}{\partial x} U_k(x,y)$.

Theorem 9

If $z(x,y,t) = \frac{\partial}{\partial y} u(x,y,t)$, then $Z_k(x,y) = \frac{\partial}{\partial y} U_k(x,y)$.

Proof: Let $z(x,y,t) = \frac{\partial}{\partial y} u(x,y,t)$, and the differential transform of $z(x,y,t)$ be $Z_k(x,y)$. Then,

$$Z_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} z(x,y,t)\right]_{t=0}$$

$$= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}(\frac{\partial}{\partial y} u(x,y,t))\right]_{t=0}$$

$$= \frac{\partial}{\partial y}\left[\frac{1}{k!}\left(\frac{\partial^k}{\partial t^k} u(x,y,t)\right)\right]_{t=0}$$

$$= \frac{\partial}{\partial y} U_k(x,y).$$

Therefore, $Z_k(x,y)=\frac{\partial}{\partial y}U_k(x,y).$

Theorem 10

If $z(x,y,t)=\frac{\partial^2}{\partial x^2}u(x,y,t),$ then $Z_k(x,y)=\frac{\partial^2}{\partial x^2}U_k(x,y).$

Proof: Let $z(x,y,t)=\frac{\partial^2}{\partial x^2}u(x,y,t),$ and the differential transform of $z(x,y,t)$ be $Z_k(x,y).$ Then,

$$\begin{aligned}
Z_k(x,y)&=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}z(x,y,t)\right]_{t=0}\\
&=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}(\frac{\partial^2}{\partial x^2}u(x,y,t))\right]_{t=0}\\
&=\frac{\partial^2}{\partial x^2}\left[\frac{1}{k!}\left(\frac{\partial^k}{\partial t^k}u(x,y,t)\right)\right]_{t=0}\\
&=\frac{\partial^2}{\partial x^2}U_k(x,y).
\end{aligned}$$

Therefore, $Z_k(x,y)=\frac{\partial^2}{\partial x^2}U_k(x,y).$

Theorem 11

If $z(x,y,t)=\frac{\partial^2}{\partial y^2}u(x,y,t),$ then $Z_k(x,y)=\frac{\partial^2}{\partial y^2}U_k(x,y).$

Proof: Let $z(x,y,t)=\frac{\partial^2}{\partial y^2}u(x,y,t),$ and the differential transform of $z(x,y,t)$ be $Z_k(x,y).$ Then,

$$\begin{aligned}
Z_k(x,y)&=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}z(x,y,t)\right]_{t=0}\\
&=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}(\frac{\partial^2}{\partial y^2}u(x,y,t))\right]_{t=0}\\
&=\frac{\partial^2}{\partial y^2}\left[\frac{1}{k!}\left(\frac{\partial^k}{\partial t^k}u(x,y,t)\right)\right]_{t=0}\\
&=\frac{\partial^2}{\partial y^2}U_k(x,y).
\end{aligned}$$

Therefore, $Z_k(x,y)=\frac{\partial^2}{\partial y^2}U_k(x,y).$

Theorem 12

If $z(x,y,t)=\frac{\partial^m}{\partial t^m}u(x,y,t)$, then $Z_k(x,y)=\frac{(k+m)!}{k!}U_{k+m}(x,y)$.

Proof: Let $z(x,y,t)=\frac{\partial^m}{\partial t^m}u(x,y,t)$ and the differential transform of $z(x,y,t)$ be $Z_k(x,y)$. Then,

$$
\begin{aligned}
Z_k(x,y) &= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}z(x,y,t)\right]_{t=0} \\
&= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\left(\frac{\partial^m}{\partial t^m}u(x,y,t)\right)\right]_{t=0} \\
&= \frac{(k+m)!}{k!(k+m)!}\left[\frac{\partial^{k+m}}{\partial t^{k+m}}u(x,y,t)\right]_{t=0} \\
&= \frac{(k+m)!}{k!}U_{k+m}(x,y).
\end{aligned}
$$

Therefore, $Z_k(x,y)=\frac{(k+m)!}{k!}U_{k+m}(x,y)$.

Theorem 13

If $z(x,y,t)=u(x,y,t)\,v(x,y,t)$, then $Z_k(x,y)=\sum_{r=0}^{k}U_r(x,y)V_{k-r}(x,y)$.

Proof: Let $z(x,y,t)=u(x,y,t)\,v(x,y,t)$ and the differential transform of $z(x,y,t)$ be $Z_k(x,y)$. Then, we have

$$
\begin{aligned}
Z_k(x,y) &= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}z(x,y,t)\right]_{t=0} \\
&= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}(u(x,y,t)\,v(x,y,t))\right]_{t=0}.
\end{aligned}
$$

Now, Leibnitz's theorem for partial derivatives of function of several variables

$$
\frac{\partial^k}{\partial t^k}(u(x,y,t)\,v(x,y,t))=\sum_{r=0}^{k}\binom{k}{r}\frac{\partial^r}{\partial t^r}u(x,y,t)\frac{\partial^{k-r}}{\partial t^{k-r}}v(x,y,t).
$$

Then, we get

$$
\begin{aligned}
Z_k(x,y) &= \frac{1}{k!}\left[\sum_{r=0}^{k}\binom{k}{r}\frac{\partial^r}{\partial t^r}u(x,y,t)\Big|_{t=0}\frac{\partial^{k-r}}{\partial t^{k-r}}v(x,y,t)\Big|_{t=0}\right] \\
&= \frac{1}{r!(k-r)!}\left[\sum_{r=0}^{k}\frac{\partial^r}{\partial t^r}u(x,y,t)\Big|_{t=0}\frac{\partial^{k-r}}{\partial t^{k-r}}v(x,y,t)\Big|_{t=0}\right] \\
&= \sum_{r=0}^{k}U_r(x,y)\,V_{k-r}(x,y).
\end{aligned}
$$

Therefore, $Z_k(x,y)=\sum_{r=0}^{k} U_r(x,y)V_{k-r}(x,y).$ □

Solving Initial Value Problems for One-Dimensional Partial Differential Equations

In this section, initial value problems for one-dimensional heat equations and wave equations are solved by using one-dimensional differential transform method.

Example 1

We consider the one-dimensional heat equation with variable coefficients as

$$u_t(x,t)+\frac{x^2}{2}u_{xx}(x,t)=0, \tag{11}$$

and the initial condition

$$u(x,0)=x^2, \tag{12}$$

where $u=u(x,t)$ is a function of the variables x and t.

Taking differential transform of (11),

$$(k+1)\,U_{k+1}(x)=-\frac{x^2}{2}\frac{\partial^2}{\partial x^2}U_k(x), \tag{13}$$

using the initial condition (12),

$$U_0(x)=x^2. \tag{14}$$

Substitution (14) into (13) and using the recurrence relation,

$$U_1(x)=-x^2,\quad U_2(x)=\frac{x^2}{2},\quad U_3(x)=-\frac{x^2}{6},\quad U_4(x)=\frac{x^2}{24},\ U_5(x)=-\frac{x^2}{120},$$

$$U_6(x)=\frac{x^2}{720},\ \ldots,\quad U_k(x)=\begin{cases}\dfrac{x^2}{k!}, & k \text{ is even},\\[2mm] -\dfrac{x^2}{k!}, & k \text{ is odd}.\end{cases}$$

Finally, the differential inverse transform of $U_k(x)$ gives: $u(x,t)=\sum_{k=0}^{\infty}U_k(x)\,t^k.$

Then, the exact solution is

$$u(x,t)=x^2e^{-t}.$$

Example 2

We consider the one-dimensional heat equation with variable coefficients as

$$u_t(x,t)-\frac{x^2}{2}u_{xx}(x,t)-2u_x(x,t)=0, \tag{15}$$

and the initial condition

$$u(x,0) = x^2, \tag{16}$$

where $u = u(x,t)$ is a function of the variables x and t.

Taking differential transform of (15),

$$(k+1)\,U_{k+1}(x) = \frac{x^2}{2}\frac{\partial^2}{\partial x^2}U_k(x) + 2\frac{\partial}{\partial x}U_k(x), \tag{17}$$

using the initial condition (16),

$$U_0(x) = x^2. \tag{18}$$

Substitution (18) into (17) and using the recurrence relation,

$$U_1(x) = x^2 + 4x, \quad U_2(x) = \frac{x^2+4x+8}{2}, \quad U_3(x) = \frac{x^2+4x+8}{6}, \quad U_4(x) = \frac{x^2+4x+8}{24},$$

$$U_5(x) = \frac{x^2+4x+8}{120}, \quad U_6(x) = \frac{x^2+4x+8}{720}, \ \ldots, U_k(x) = \frac{x^2+4x+8}{k!}.$$

Finally, the differential inverse transform of $U_k(x)$ gives:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)\,t^k = (x^2+4x+8)\sum_{k=0}^{\infty}\frac{t^k}{k!}.$$

Then, the exact solution is

$$u(x,t) = (x^2+4x+8)\,e^t.$$

Example 3

We consider the linear Klein-Gordon equation in the form

$$u_{tt}(x,t) - u_{xx}(x,t) - u(x,t) = 0, \tag{19}$$

and the initial conditions

$$u(x,0) = 1 + \cos x, u_t(x,0) = 0, \tag{20}$$

where $u = u(x,t)$ is a function of the variables x and t.

Taking differential transform of (19),

$$(k+1)(k+2)U_{k+2}(x) = \frac{\partial^2}{\partial x^2}U_k(x) + U_k(x), \tag{21}$$

using the initial condition (20),

$$U_0(x) = 1 + \cos x, U_1(x) = 0. \tag{22}$$

Substitution (22) into (21) and using the recurrence relation,

$$U_k(x) = 0,\ k = 1,3,5,\ldots.$$

By applying the k values are $k = 2,4,6,\ldots,$

$$U_2(x)=\frac{1}{2},\quad U_4(x)=\frac{1}{24},\quad U_6(x)=\frac{1}{720},\ \ldots,\ U_k(x)=\frac{1}{k!}.$$

Finally, the differential inverse transform of $U_k(x)$ gives:

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)\,t^k=(1+\cos x)\ +(\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+\ldots).$$

Then, the exact solution is $u(x,t)=\cos x+\cosh t.$

Example 4

We consider the one-dimensional wave equation with variable coefficients as

$$u_{tt}(x,t)+\frac{x^2}{2}u_{xx}(x,t)=0, \tag{23}$$

and the initial conditions

$$u(x,0)=x, u_t(x,0)=x^2, \tag{24}$$

where $u=u(x,t)$ is a function of the variables x and t.

Taking differential transform of (23),

$$(k+1)(k+2)\,U_{k+2}(x)=-\frac{x^2}{2}\frac{\partial^2}{\partial x^2}U_k(x), \tag{25}$$

using the initial condition (24),

$$U_0(x)=x, U_1(x)=x^2. \tag{26}$$

Substitution (26) into (25) and using the recurrence relation,

$$U_k(x)=0,\ k=2,4,6,\ldots.$$

By applying the k values are $k=1,3,5,\ldots,$

$$U_3(x)=-\frac{x^2}{6},\quad U_5(x)=\frac{x^2}{120},\quad U_7(x)=-\frac{x^2}{5040},\ \ldots,$$

$$U_k(x)=\begin{cases}(-1)^k\,\dfrac{x^2}{k!}\ , & k=3,7,11,\ldots,\\[2ex] (-1)^{k+1}\,\dfrac{x^2}{k!}, & k=5,9,13,\ldots.\end{cases}$$

Finally, the differential inverse transform of $U_k(x)$ gives:

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)\,t^k=x+x^2\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\ldots+\frac{t^k}{k!}\right).$$

Then, the exact solution is

$$u(x,t) = x + x^2 \sin t.$$

Example 5

We consider the one-dimensional wave equation with variable coefficients as

$$u_{tt}(x,t) - \frac{x^2}{2} u_{xx}(x,t) - u_x(x,t) = 0, \tag{27}$$

and the initial conditions

$$u(x,0) = 1, u_t(x,0) = x^2, \tag{28}$$

where $u = u(x,t)$ is a function of the variables x and t.

Taking differential transform of (27),

$$(k+1)(k+2)\,U_{k+2}(x) = \frac{x^2}{2}\frac{\partial^2}{\partial x^2}U_k(x) + \frac{\partial}{\partial x}U_k(x), \tag{29}$$

using the initial condition (28),

$$U_0(x) = 1, U_1(x) = x^2. \tag{30}$$

Substitution (30) into (29) and using the recurrence relation,

$$U_k(x) = 0,\ k = 2,4,6,\ldots.$$

By applying the k values are $k = 1,3,5,\ldots,$

$$U_3(x) = \frac{x^2+2x}{6}, U_5(x) = \frac{x^2+2x+2}{120}, U_7(x) = \frac{x^2+2x+2}{5040}, \ldots,$$

$$U_k(x) = \frac{x^2+2x+2}{k!}.$$

Finally, the differential inverse transform of $U_k(x)$ gives:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)\,t^k = 1 + (x^2+2x+2)\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \ldots + \frac{t^k}{k!}\right).$$

Then, the exact solution is

$$u(x,t) = 1 + (x^2+2x+2)\sinh t.$$

Solving Initial Value Problems for Two-Dimensional Partial Differential Equations

In this section, initial value problems for two-dimensional heat equations and wave equations are solved by using one-dimensional differential transform method.

Example 6

We consider the two-dimensional heat equation with variable coefficients as

$$u_t(x,y,t)-\frac{y^2}{2}u_{xx}(x,y,t)-\frac{x^2}{2}u_{yy}(x,y,t)=0, \tag{31}$$

and the initial condition

$$u(x,y,0)=x^2, \tag{32}$$

where $u=u(x,t)$ is a function of the variables x and t.

Taking differential transform of (31),

$$(k+1)U_{k+1}(x,y)=\frac{y^2}{2}\frac{\partial^2}{\partial x^2}U_k(x,y)+\frac{x^2}{2}\frac{\partial^2}{\partial y^2}U_k(x,y), \tag{33}$$

using the initial condition (32),

$$U_0(x,y)=x^2. \tag{34}$$

Substitution (34) into (33) and using the recurrence relation,

$$U_1(x,y)=y^2,\quad U_2(x,y)=\frac{x^2}{2},\quad U_3(x,y)=\frac{y^2}{6},\ U_4(x,y)=\frac{x^2}{24},\ U_5(x,y)=\frac{y^2}{120},$$

$$U_6(x,y)=\frac{x^2}{720},\ U_7(x,y)=\frac{y^2}{5040},\ \ldots,\ U_k(x,y)=\begin{cases}\dfrac{x^2}{k!}, & k\,\text{is even},\\[2mm] \dfrac{y^2}{k!}, & k\,\text{is odd}.\end{cases}$$

Finally, the differential inverse transform of $U_k(x,y)$ gives:

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)\,t^k=x^2\sum_{k=0,2,4}^{\infty}\frac{t^k}{k!}+y^2\sum_{k=1,3,5}^{\infty}\frac{t^k}{k!}.$$

Then, the exact solution is

$$u(x,y,t)=x^2(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots)+y^2(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\ldots)=x^2\cosh t+y^2\sinh t.$$

Example 7

We consider the two-dimensional heat equation with variable coefficients as

$$u_t(x,y,t)+\frac{y^2}{2}u_{xx}(x,y,t)+\frac{x^2}{2}u_{yy}(x,y,t)+u_x(x,y,t)+u_y(x,y,t)=0, \tag{35}$$

and the initial condition

$$u(x,y,0)=y^2, \tag{36}$$

where $u=u(x,t)$ is a function of the variables x and t.

Taking differential transform of (35),

$$(k+1)U_{k+1}(x,y) = -\frac{y^2}{2}\frac{\partial^2}{\partial x^2}U_k(x,y) - \frac{x^2}{2}\frac{\partial^2}{\partial y^2}U_k(x,y) - \frac{\partial}{\partial x}U_k(x,y) - \frac{\partial}{\partial y}U_k(x,y), \tag{37}$$

using the initial condition (36),

$$U_0(x,y) = y^2. \tag{38}$$

Substitution (38) into (37) and using the recurrence relation,

$$U_1(x,y) = -(x^2+2y),\quad U_2(x,y) = \frac{y^2+2x+2}{2},\quad U_3(x,y) = -(\frac{x^2+2y+2}{6}),$$

$$U_4(x,y) = \frac{y^2+2x+2}{24},\quad U_5(x) = -(\frac{x^2+2y+2}{120}),\quad U_6(x) = \frac{y^2+2x+2}{720},\ \ldots,$$

$$U_k(x,y) = \begin{cases} \dfrac{y^2+2x+2}{k!}, & k \text{ is even}, \\ -\dfrac{(x^2+2y+2)}{k!}, & k \text{ is odd}. \end{cases}$$

Finally, the differential inverse transform of $U_k(x,y)$ gives:

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)\,t^k = (y^2+2x+2)\sum_{k=0}^{\infty}\frac{t^k}{k!} - (x^2+2y+2)\sum_{k=0}^{\infty}\frac{t^k}{k!}.$$

Then, the exact solution is

$$u(x,y,t) = (y^2+2x+2)(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots) - (x^2+2y+2)(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\ldots)$$

$$= (y^2+2x+2)\cosh t - (x^2+2y+2)\sinh t.$$

Example 8

We consider the two-dimensional wave equation with constant coefficients as

$$u_{tt}(x,y,t) - u_{xx}(x,y,t) - u_{yy}(x,y,t) - u(x,y,t) = 0, \tag{39}$$

and the initial conditions

$$u(x,y,0) = 1+\cos x, u_t(x,0) = 0, \tag{40}$$

where $u = u(x,t)$ is a function of the variables x and t.

Taking differential transform of (39),

$$(k+1)(k+2)U_{k+2}(x,y) = \frac{\partial^2}{\partial x^2}U_k(x,y) + \frac{\partial^2}{\partial y^2}U_k(x,y) + U_k(x,y), \tag{41}$$

using the initial condition (40),

$$U_0(x,y) = 1+\cos x, U_1(x,y) = 0. \tag{42}$$

Substitution (42) into (41) and using the recurrence relation,

$$U_k(x,y)=0,\ k=1,3,5,\ldots.$$

By applying the k values are $k=2,4,6,\ldots,$

$$U_2(x,y)=\frac{1}{2},\quad U_4(x,y)=\frac{1}{24},\quad U_6(x,y)=\frac{1}{720},\ \ldots,\ U_k(x,y)=\frac{1}{k!}.$$

Finally, the differential inverse transform of $U_k(x)$ gives:

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)\,t^k=(1+\cos x)+(\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+\ldots).$$

Then, the exact solution is

$$u(x,y,t)=\cos x+\cosh t.$$

Example 9

We consider the two-dimensional wave equation with variable coefficients as

$$u_{tt}(x,y,t)-\frac{x^2}{12}u_{xx}(x,y,t)-\frac{y^2}{12}u_{yy}(x,y,t)=0, \tag{43}$$

and the initial conditions

$$u(x,y,0)=x^4,\ u_t(x,y,0)=y^4, \tag{44}$$

where $u=u(x,t)$ is a function of the variables x and t.

Taking differential transform of (43),

$$(k+1)(k+2)U_{k+2}(x,y)=\frac{x^2}{12}\frac{\partial^2}{\partial x^2}U_k(x,y)+\frac{y^2}{12}\frac{\partial^2}{\partial y^2}U_k(x,y), \tag{45}$$

using the initial condition (44),

$$U_0(x,y)=x^4,\ U_1(x,y)=y^4. \tag{46}$$

Substitution (46) into (45) and using the recurrence relation,

$$U_2(x,y)=\frac{x^4}{2},\quad U_3(x,y)=\frac{y^4}{6},\quad U_4(x,y)=\frac{x^4}{24},\quad U_5(x,y)=\frac{y^4}{120},$$

$$U_6(x,y)=\frac{x^4}{720},\ \ldots,\ U_k(x,y)=\begin{cases}\dfrac{x^4}{k!}, & k\ \text{is even},\\[2mm] \dfrac{y^4}{k!}, & k\ \text{is odd}.\end{cases}$$

Finally, the differential inverse transform of $U_k(x,y)$ gives:

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)\,t^k=x^4\sum_{k=0,2,4}^{\infty}U_k(x,y)\,t^k+y^4\sum_{k=1,3,5}^{\infty}U_k(x,y)\,t^k.$$

Then, the exact solution is

$$u(x,y,t)=x^4(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots)+y^4(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\ldots)=x^4\cosh t+y^4\sinh t.$$

Conclusion

In this paper, we have studied one-dimensional and two-dimensional heat and wave equations with the help of differential transform method. The differential transform method has been successful, applied for solving linear and homogeneous partial differential equations with constant coefficients and variable coefficients. We conclude that differential transform method can be extended to solve many partial differential equations with constant coefficients and variable coefficients which arise in physical and engineering applications.

Acknowledgements

We would like to thank Rector Dr. Tin Tun and Pro-Rectors, University of Mandalay, for giving us their kind permission to undertake this paper. I would also like to thank Head and Professor Dr. Ko Ko Lwin, Department of Mathematics, University of Mandalay, for his permission and support concerning this research paper. I am grateful to my supervisor Dr. Khin Than Sint, Professor, Department of Mathematics, University of Mandalay, for her advice and suggestion to implement this kind of research paper.

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